Lecture 6

- General Overview of Non-Linear Programming
- Portfolio Optimization - Part II
  - The Efficient Frontier and Correlation
  - An Example with Real Data
    - Adjusting the data to match forecasts
  - Adding a constraint on the number of securities in an optimal portfolio
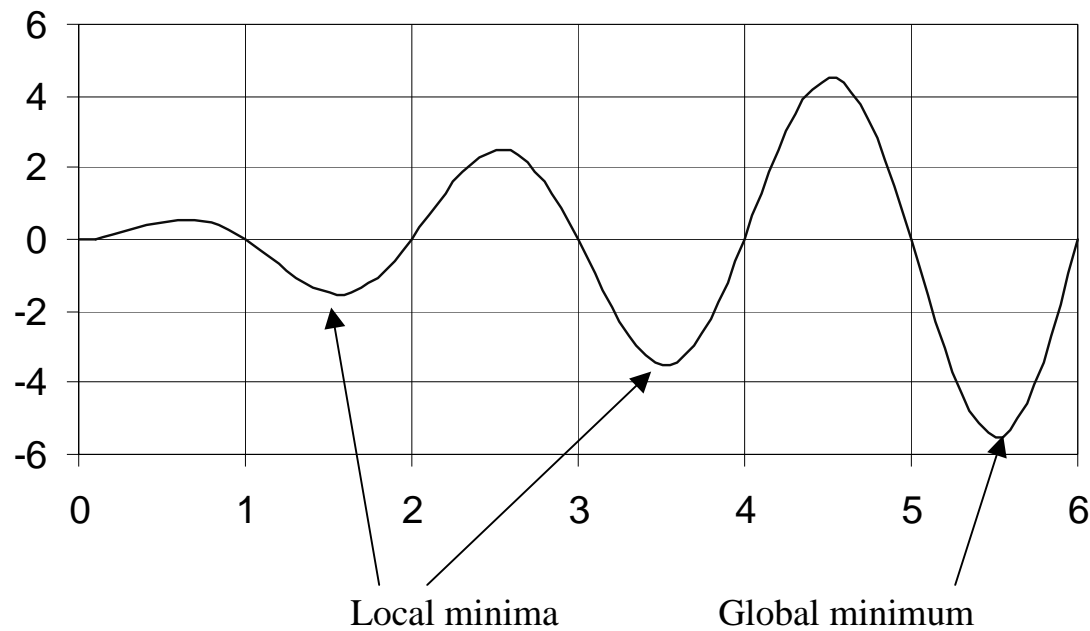- Summary and Preparation for next class

Nonlinear Programming

$$\min_x \ y = x \sin(\pi x)$$

subject to:

(Upper bound) $x \leq 6$

(Lower bound) $x \geq 0$

Graph of $x \sin(\pi x)$ vs. x

Local minima   Global minimum

Nonlinear Programming (continued)

○ Starting from $x = 0$:   the optimizer converges to

$$(1)\ \ x^* = 1.56,\ y^* = -1.53.$$

Starting from $x = 3$:   the optimizer converges to

$$(2)\ \ x^* = 3.53,\ y^* = -3.51.$$

Starting from $x = 5$:   the optimizer converges to

$$(3)\ \ x^* = 5.52,\ y^* = -5.51.$$

The solution returned by the optimizer depends on the starting point.

(1) and (2) are *local minima* of the nonlinear program.

(3) is the *global minimum*, i.e., it is the true optimal solution.

○ In general, optimizers are not guaranteed to give global optimal solutions to nonlinear programs.

Nonlinear Programming (continued)

○ Not all nonlinear programs have local optima. In fact, mean-variance models are well-behaved: the only local optimum is also a global optimum. A sample graph of portfolio standard deviation versus portfolio weights x_1 and x_2 is given below. For mean-variance problems, the optimizer should return the correct global-optimal solution.
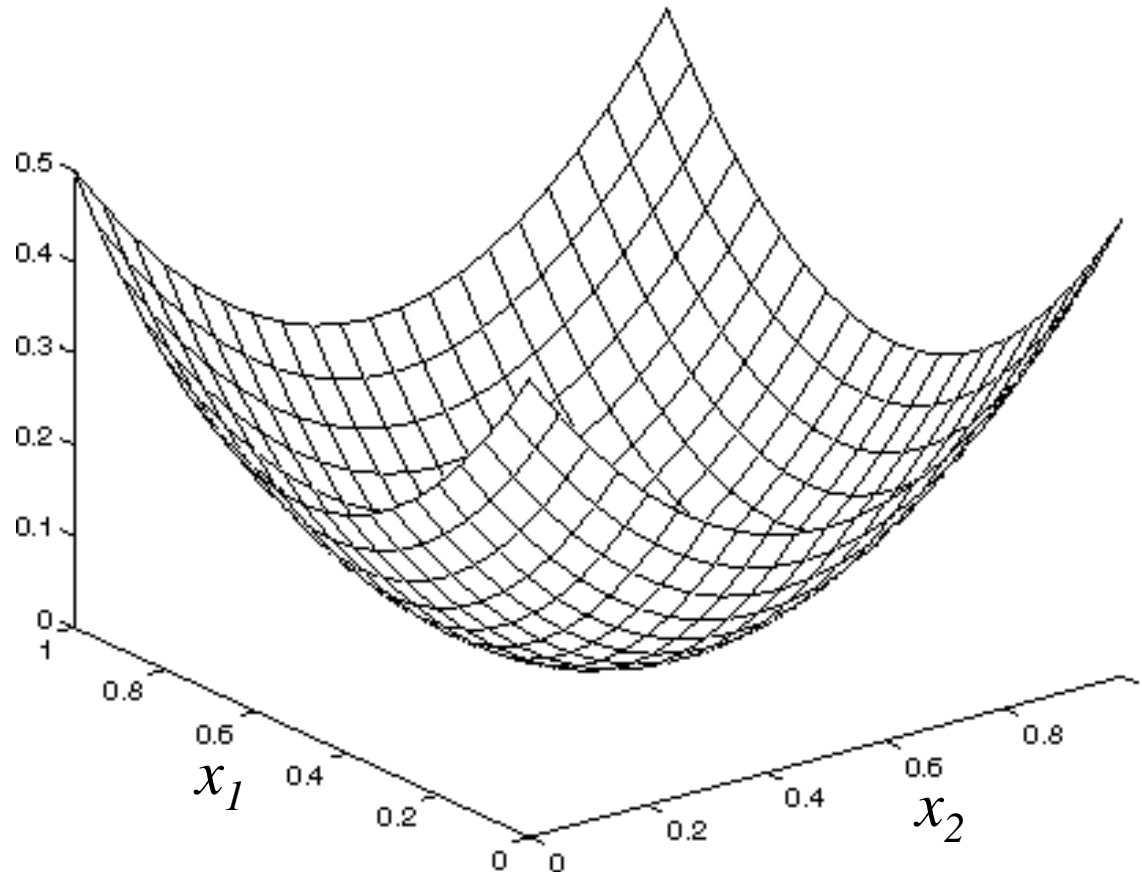
The Efficient Frontier and Correlation:
Examples with Two Securities

○ Suppose you have two securities that are highly correlated.

○ Let's say there are 10 equal-probability scenarios.

○ Data:

Scenario	Scenario returns r(i,j) by Security		
	A	B	
1	5.33	4.06	
2	2.54	1.26	
3	1.43	1.38	
4	4.55	3.29	
5	2.44	1.11	
6	-0.22	-0.34	
7	-4.44	-4.87	
8	-0.22	-1.31	
9	0.58	0.78	
10	-0.92	-1.12	
Average	**1.11**	**0.42**	
StdDev	**2.69**	**2.40**	
		Correlation	
Sec A vs. Sec B		**0.98**	

High Positive Correlation

○ For these two highly (positively) correlated securities, the efficient frontier is very nearly a straight line, from security A (representing a 100% investment in A) to security B (representing a 100% investment in security B).

○ There are very little or no benefits to "diversification" in this case.

Efficient Frontier for a Portfolio of Two Stocks with Correlation +0.98
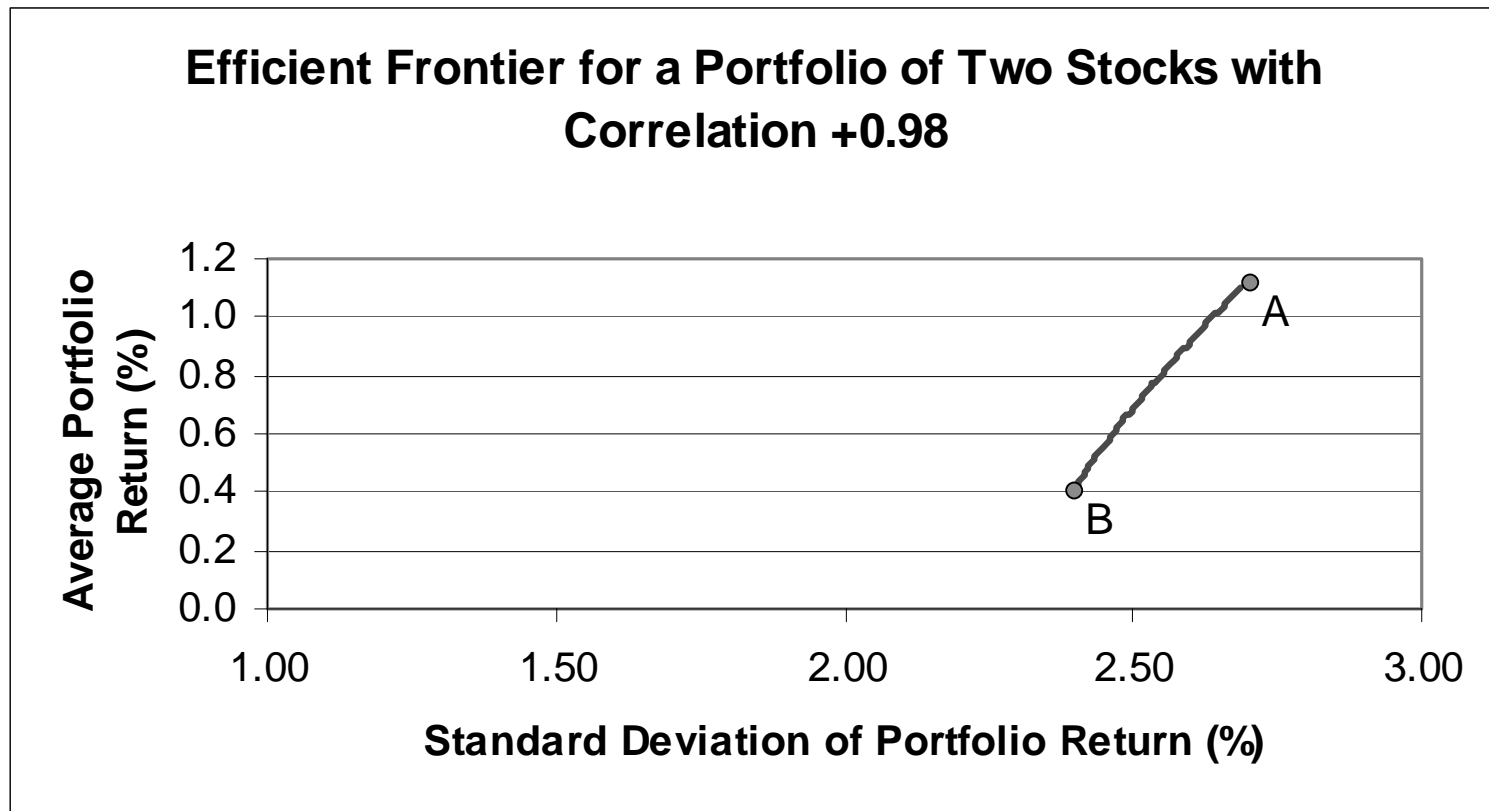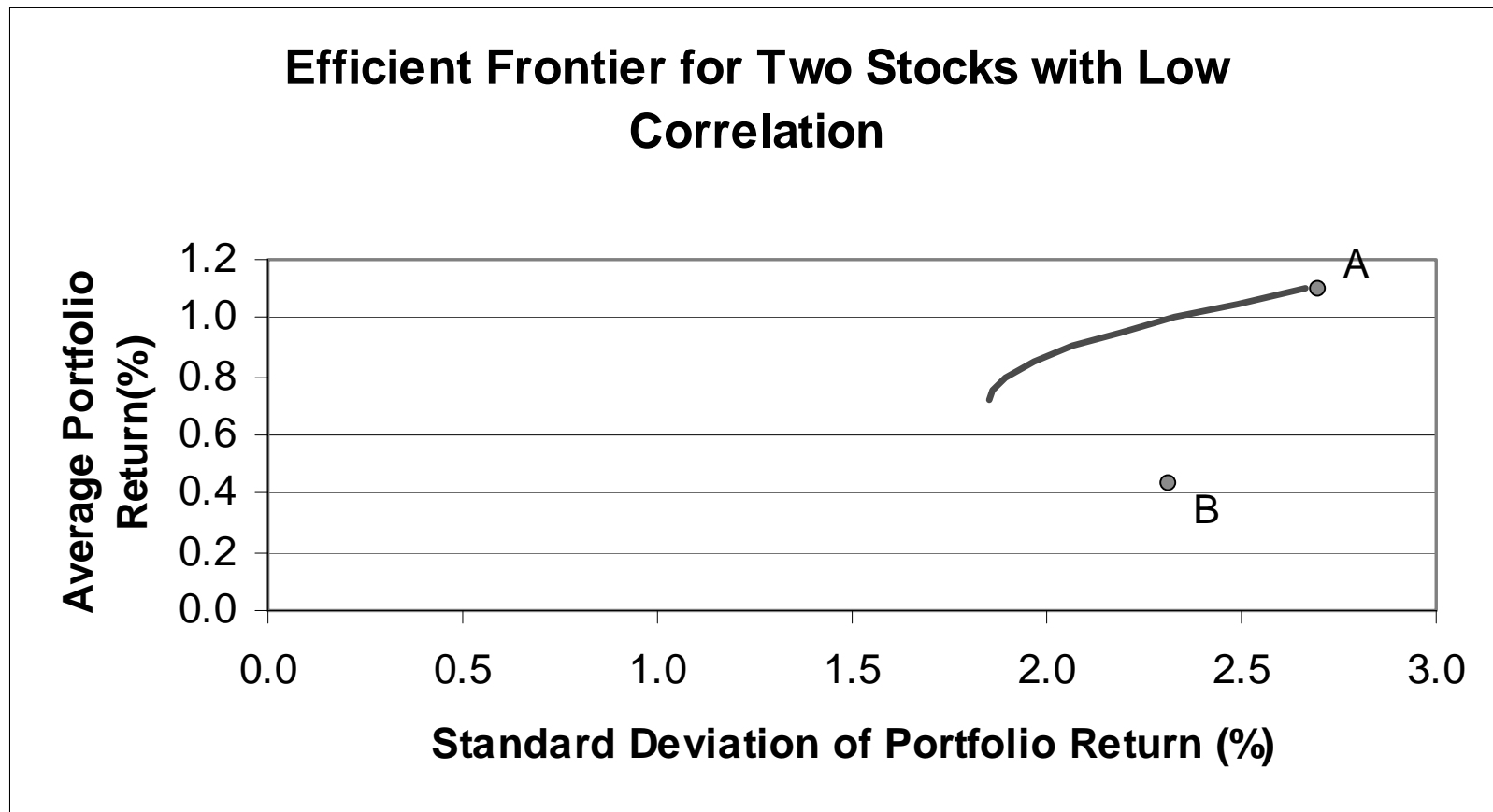
Low Correlation (Uncorrelated returns)

○ Now suppose you have two securities that are not very correlated (say correlation close to 0).

○ Again, let's say there are 10 scenarios (with equal probabilities).

○ Data:

Scenario	Scenario returns r(i,j) by Security		
	A	B	
1	5.33	3.47	
2	2.54	-2.74	
3	1.43	-1.27	
4	4.55	-2.53	
5	2.44	3.22	
6	-0.22	1.33	
7	-4.44	-1.22	
8	-0.22	-0.80	
9	0.58	3.41	
10	-0.92	1.52	
Average	**1.11**	**0.44**	
StdDev	**2.69**	**2.32**	
		Correlation	
Sec A vs. Sec B		**0.11**	

Low Correlation

○ The efficient frontier is a curve extending to the left of both A and B.

○ This illustrates the benefits of diversification.

**Efficient Frontier for Two Stocks with Low Correlation**

Very Negative Correlation

○ Now suppose you have two securities that are highly negatively correlated (say correlation close to -1).

○ Again, let's say there are 10 scenarios (with equal probabilities).

○ Data:

Scenario	Scenario returns r(i,j) by Security		
	A	**B**	
1	5.33	-4.14	
2	2.54	-1.51	
3	1.43	0.64	
4	4.55	-1.48	
5	2.44	-0.41	
6	-0.22	1.21	
7	-4.44	4.45	
8	-0.22	1.26	
9	0.58	2.44	
10	-0.92	1.91	
Average	**1.11**	**0.44**	
StdDev	**2.69**	**2.30**	
		Correlation	
Sec A vs. Sec B		**-0.94**	

Very Negative Correlation

○ The efficient frontier is almost a straight line.

○ It is possible to construct a nearly risk-less portfolio with these 2 stocks.

Efficient Frontier for Two Stocks with -0.94 Correlation
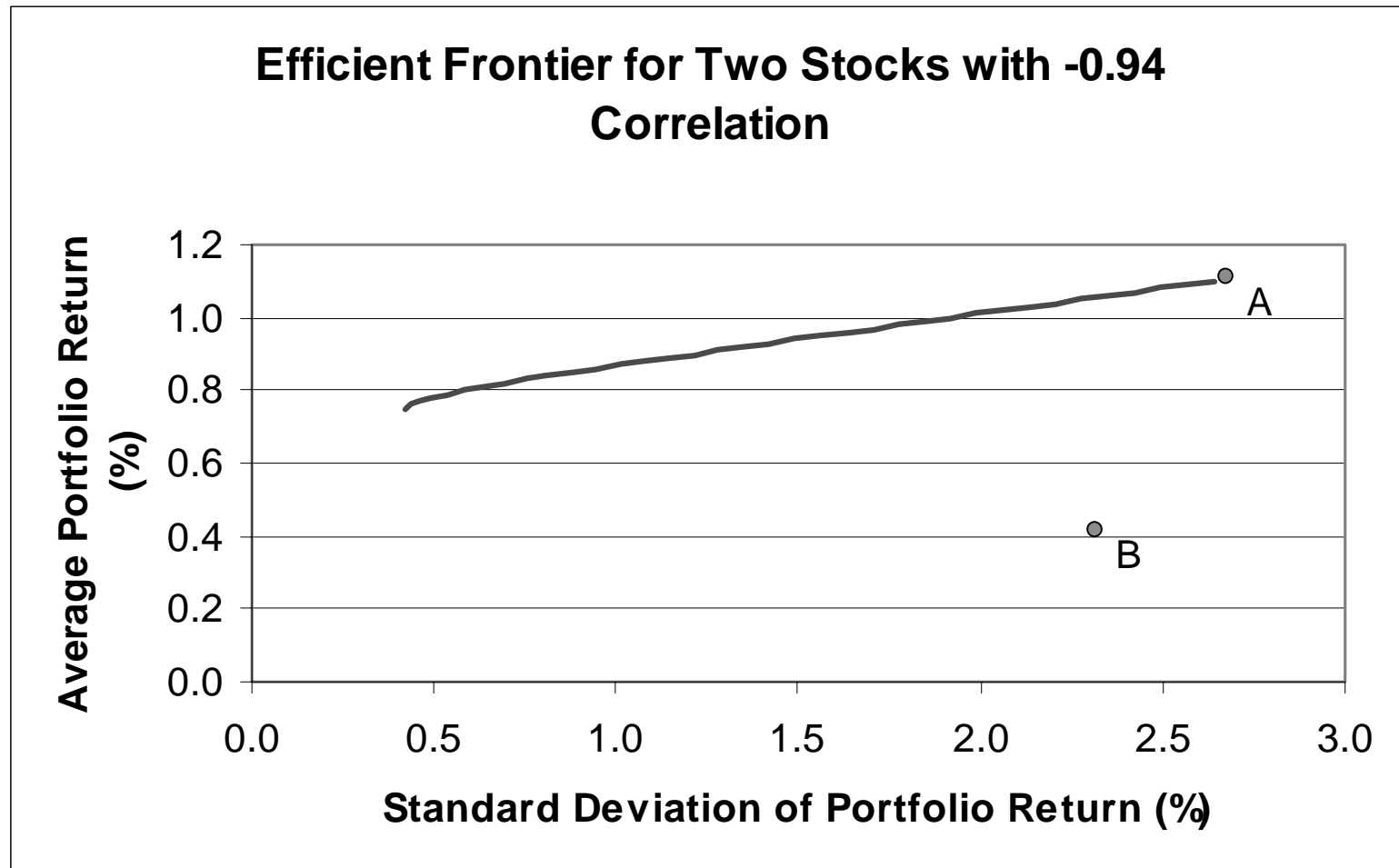
Portfolio Optimization (continued)

○ Using historical stock-return data, it is possible to develop meaningful scenarios. Consider the following ten stocks: Apple, GM, IBM, Merck, Ford, J&J, P&G, Sun, Intel and Microsoft. We list their monthly returns during the period from January 1996 to December 1997 (24 months).

Monthly Returns in %

	SUN	MSFT	GM	IBM	APPLE	P&G	J&J	MERCK	FORD	INTEL
Jan-96	0.8%	5.4%	-0.5%	18.7%	-13.3%	1.2%	12.3%	6.9%	2.2%	-2.7%
Feb-96	14.1%	6.7%	-2.6%	13.0%	-0.5%	-2.4%	-2.6%	-5.5%	5.9%	6.5%
Mar-96	-16.7%	4.5%	3.9%	-9.3%	-10.7%	3.4%	-1.3%	-6.0%	10.0%	-3.3%
Apr-96	24.0%	9.8%	1.9%	-3.1%	-0.8%	-0.3%	0.3%	-2.8%	4.4%	19.1%
May-96	15.4%	4.9%	1.6%	-0.9%	7.2%	4.0%	5.3%	6.8%	1.7%	11.4%
Jun-96	-6.0%	1.2%	-5.0%	-7.3%	-19.6%	3.1%	1.7%	0.0%	-11.3%	-2.7%
Jul-96	-7.2%	-1.9%	-6.9%	8.6%	4.8%	-1.5%	-3.5%	-0.6%	0.0%	2.3%
Aug-96	-0.5%	3.9%	1.8%	6.4%	10.2%	-0.4%	3.1%	2.1%	3.5%	6.2%
Sep-96	14.3%	7.7%	-3.3%	8.9%	-8.5%	9.7%	4.1%	7.2%	-6.7%	19.6%
Oct-96	-1.8%	4.1%	11.7%	3.6%	3.7%	1.5%	-3.9%	5.0%	0.0%	15.1%
Nov-96	-4.5%	14.3%	7.5%	23.5%	4.9%	9.8%	8.1%	12.4%	4.8%	15.5%
Dec-96	-11.8%	5.3%	-3.3%	-4.9%	-13.5%	-1.0%	-6.6%	-4.1%	-1.5%	3.2%
Jan-97	23.6%	23.4%	5.8%	3.5%	-20.4%	7.4%	16.1%	13.8%	-0.4%	23.9%
Feb-97	-2.8%	-4.4%	-1.9%	-8.4%	-2.3%	3.9%	-0.4%	1.7%	2.3%	-12.6%
Mar-97	-6.5%	-6.0%	-4.3%	-4.5%	12.3%	-4.5%	-8.0%	-8.5%	-4.6%	-1.9%
Apr-97	-0.2%	32.5%	4.5%	16.9%	-6.8%	9.6%	15.6%	7.3%	10.8%	10.1%
May-97	11.9%	2.1%	-0.9%	7.8%	-2.2%	9.6%	-1.8%	-0.6%	7.9%	-1.1%
Jun-97	15.4%	1.9%	-2.8%	4.3%	-14.3%	2.4%	7.3%	13.8%	1.3%	-6.4%
Jul-97	22.8%	12.0%	11.0%	17.2%	22.8%	7.7%	-3.5%	1.5%	7.6%	29.5%
Aug-97	5.6%	-6.6%	1.4%	-4.1%	24.3%	-12.5%	-8.8%	-11.6%	5.2%	0.3%
Sep-97	-3.0%	0.1%	6.7%	4.6%	-0.3%	3.8%	1.8%	8.9%	4.9%	0.2%
Oct-97	-26.8%	-1.7%	-4.1%	-7.1%	-21.5%	-1.5%	-0.5%	-10.7%	-3.2%	-16.6%
Nov-97	5.1%	8.8%	-5.1%	11.2%	4.2%	12.0%	9.7%	6.2%	-1.6%	0.8%
Dec-97	10.8%	-8.7%	5.8%	-4.5%	-26.1%	4.8%	4.7%	11.8%	12.9%	-9.5%

Portfolio Optimization (continued)

❍ We expand the spreadsheet model to include these ten stocks and the 24 scenarios.

❍ We assign a probability of 1/24=0.04167% to each scenario.

❍ The rest of the spreadsheet is as in the previous example.

❍ Some questions:

▸ For each stock, we can calculate the mean and standard deviation of the return in our model:

	SUN	MSFT	GM	IBM	APPLE	P&G	J&J	MERCK	FORD	INTEL
Mean=	3.2%	5.0%	1.0%	3.9%	-2.8%	2.9%	2.0%	2.3%	2.3%	4.5%
Stnd. Dev.=	12.8%	9.0%	5.1%	9.3%	13.0%	5.5%	6.7%	7.4%	5.5%	11.4%

▸ Are these accurate reflections of returns in the coming month? How about correlations between the stocks? Are they reflected here?

Do the historical means accurately reflect the mean returns in the coming month?

○ Suppose we do not believe that the historical mean returns are an accurate reflection of the returns that might be expected in the coming month.

○ History is not likely to repeat itself exactly - so we revise the mean return estimates, but assume continuation of past volatility levels and correlation.

○ One method of doing this is using CAPM (Capital Asset Pricing Model).

○ To do that, we need the following:

▸ r_m = an estimate of market's expected return in the coming month

▸ r_f = an estimate of the risk-free rate over the coming month

▸ For each security, β_i = security i's beta.

○ Let's say the following: r_m=15%/12=1.25% and r_f=6%/12=0.5%.

○ The betas of the ten securities are:

SUN	MSFT	GM	IBM	APPLE	P&G	J&J	MERCK	FORD	INTEL
1.41	1.37	1.13	1.08	0.52	0.81	0.94	1.01	1.04	1.25

○ How do we adjust the means, and the optimization model?

Adjusting the historical means (cont.)

❍ The new means are calculated as follows (from CAPM):

▸ Let μ be the average return of security A, with β = beta of A.

▸ Then, μ is calculated as follows:

$$\mu = r_f + \beta \, (r_m\text{-}r_f)$$

❍ This means that we can determine estimated average returns for each of the securities.

❍ We get:

	SUN	MSFT	GM	IBM	APPLE	P&G	J&J	MERCK	FORD	INTEL
Historical Means=	3.2%	5.0%	1.0%	3.9%	-2.8%	2.9%	2.0%	2.3%	2.3%	4.5%
New Estimated Means=	1.6%	1.5%	1.3%	1.3%	0.9%	1.1%	1.2%	1.3%	1.3%	1.4%

❍ It is easy then to adjust the means to these numbers. However, remember our optimization model does not read the "mean return" cell, it works off of the scenarios. So we must adjust the scenario returns so that their mean matches these adjusted means.

Adjusting the historical means (cont.)

❍ How do we then adjust the historical scenario returns to match the new adjusted means?

❍ The simplest way to do this is to shift all the scenario outcomes by the required amount. To understand this, take one of the securities, say MSFT.

  ‣ Its historical monthly mean return was 5.0%.

  ‣ Our revised estimate is 1.5%, or 3.5% lower.

  ‣ So subtract 3.5% from each of MSFT's historical scenario returns.

  ‣ Repeat this for each security.

❍ What do we do about the standard deviations? Are the correlations intact?

❍ Now we can run the model.

❍ We want to determine the minimum-risk (i.e., minimum-standard-deviation) portfolio that invests 100% in these stocks and achieves a mean portfolio return of at least 1.35% during the next month. How diversified is this portfolio?

Optimized Spreadsheet

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	TEN-STOCKS.XLS			**Investment Non-Linear Program**											
2									Sum of Portfolio Weights						
3			Avg. Portfolio	Portfolio		Portfolio Weights x(j)				100%	=	100%			
4			Return	Stnd. Dev.		**SUN**	**MSFT**	**GM**	**IBM**	**APPLE**	**P&G**	**J&J**	**MERCK**	**FORD**	**INTEL**
5			1.35%	4.43%		5.1%	13.0%	**46.5%**	0.0%	0.0%	0.0%	10.5%	0.0%	24.8%	0.0%
6			>=			**SUN**	**MSFT**	**GM**	**IBM**	**APPLE**	**P&G**	**J&J**	**MERCK**	**FORD**	**INTEL**
7		Min Return	1.35%	Mean=		1.6%	1.5%	1.3%	1.3%	0.9%	1.1%	1.2%	1.3%	1.3%	1.4%
8				Stnd. Dev.=		12.8%	9.0%	5.1%	9.3%	13.0%	5.5%	6.7%	7.4%	5.5%	11.4%
9															
10				Adjusted Mean=		1.6%	1.5%	1.3%	1.3%	0.9%	1.1%	1.2%	1.3%	1.3%	1.4%
11															
12		Scen-	Proba-	Portfolio Return		Security Returns by Scenario									
13		ario	bilities	by Scenario		**SUN**	**MSFT**	**GM**	**IBM**	**APPLE**	**P&G**	**J&J**	**MERCK**	**FORD**	**INTEL**
14		1	1/24	1.7%		-0.8%	2.0%	-0.1%	16.1%	-9.7%	-0.6%	11.5%	5.8%	1.1%	-5.7%
15		2	1/24	0.9%		12.5%	3.2%	-2.2%	10.4%	3.2%	-4.2%	-3.4%	-6.6%	4.9%	3.5%
16		3	1/24	3.2%		-18.3%	1.1%	4.3%	-11.9%	-7.0%	1.5%	-2.2%	-7.1%	8.9%	-6.3%
17		4	1/24	3.8%		22.4%	6.4%	2.3%	-5.8%	2.9%	-2.1%	-0.6%	-3.8%	3.3%	16.1%
18		5	1/24	2.5%		13.8%	1.4%	2.0%	-3.5%	10.8%	2.2%	4.4%	5.8%	0.7%	8.4%
19		6	1/24	-5.8%		-7.6%	-2.3%	-4.6%	-9.9%	-16.0%	1.3%	0.8%	-1.0%	-12.4%	-5.8%
20		7	1/24	-4.9%		-8.8%	-5.3%	-6.5%	6.0%	8.4%	-3.3%	-4.4%	-1.6%	-1.1%	-0.7%
21		8	1/24	1.8%		-2.1%	0.5%	2.2%	3.8%	13.9%	-2.2%	2.3%	1.1%	2.4%	3.2%
22		9	1/24	-1.7%		12.6%	4.2%	-2.9%	6.2%	-4.9%	7.9%	3.2%	6.2%	-7.8%	16.6%
23		10	1/24	4.8%		-3.4%	0.6%	12.1%	1.0%	7.3%	-0.3%	-4.7%	3.9%	-1.1%	12.1%
24		11	1/24	6.4%		-6.1%	10.9%	7.8%	20.9%	8.5%	8.0%	7.3%	11.3%	3.7%	12.4%
25		12	1/24	-3.2%		-13.4%	1.9%	-2.9%	-7.6%	-9.8%	-2.8%	-7.4%	-5.1%	-2.6%	0.2%
26		13	1/24	7.9%		22.0%	20.0%	6.2%	0.9%	-16.7%	5.6%	15.2%	12.8%	-1.4%	20.9%
27		14	1/24	-1.8%		-4.4%	-7.9%	-1.5%	-11.0%	1.4%	2.1%	-1.3%	0.6%	1.3%	-15.6%
28		15	1/24	-5.8%		-8.1%	-9.4%	-3.9%	-7.1%	16.0%	-6.3%	-8.9%	-9.6%	-5.6%	-5.0%
29		16	1/24	9.9%		-1.8%	29.1%	4.9%	14.3%	-3.2%	7.8%	14.8%	6.2%	9.7%	7.0%
30		17	1/24	1.5%		10.3%	-1.4%	-0.5%	5.2%	1.4%	7.8%	-2.7%	-1.6%	6.9%	-4.1%
31		18	1/24	0.1%		13.8%	-1.5%	-2.4%	1.7%	-10.6%	0.6%	6.5%	12.8%	0.3%	-9.4%
32		19	1/24	8.6%		21.1%	8.5%	11.4%	14.6%	26.5%	5.9%	-4.3%	0.5%	6.5%	26.5%
33		20	1/24	-0.2%		4.0%	-10.0%	1.8%	-6.8%	27.9%	-14.3%	-9.6%	-12.6%	4.1%	-2.7%
34		21	1/24	3.7%		-4.6%	-3.3%	7.1%	1.9%	3.4%	2.0%	0.9%	7.8%	3.9%	-2.8%
35		22	1/24	-5.1%		-28.4%	-5.2%	-3.7%	-9.7%	-17.8%	-3.3%	-1.4%	-11.7%	-4.2%	-19.6%
36		23	1/24	-1.0%		3.5%	5.4%	-4.7%	8.6%	7.9%	10.2%	8.9%	5.2%	-2.6%	-2.2%
37		24	1/24	5.1%		9.2%	-12.1%	6.2%	-7.1%	-22.4%	3.0%	3.8%	10.8%	11.9%	-12.5%

Portfolio Optimization Solver Parameters

Solver Parameters   `?` `X`

Set Target Cell: `$D$5`   **Solve**

Equal To: ○ Max ⦿ Min ○ Value of: `0`   **Close**

By Changing Cells:

`$F$5:$O$5`   **Guess**

Subject to the Constraints:

```
$C$5 >= $C$7
$J$3 = $L$3
```

Add

Change

Delete

Options

Reset All

Help

The solver parameters dialog box

Portfolio Optimization (continued)

○ As can be seen from the optimized spreadsheet, the model suggests to invest in positive quantities in these five stocks:

    ▸ **SUN   MSFT   GM   J&J   FORD**

      5.1%   13.0%   46.5%   10.5%   24.8%

    ▸ It invests nothing in IBM, APPLE, P&G, Merck or Intel.

○ The average portfolio return is: 1.35%.

○ The standard deviation (SD) of the portfolio return is: 4.43%.

○ Comments:

    ▸ The portfolio is heavily invested (46.5+24.8=71.3%) in the two "safest" stocks (Ford and GM), as measured by *SD*.

    ▸ The portfolio is invested in J&J (with average return 1.2%) but not in Intel (average return 1.4%).

    ▸ Our average portfolio return is 1.35%, which is exactly the minimum average return we had specified.

The Efficient Frontier

❍ Suppose we want to vary the minimum mean return (δ) of the portfolio.

❍ Using SolverTable, we can vary δ and trace out an efficient frontier.

❍ Consider minimizing SD and varying δ from 1.18% to 1.6% in increments of 0.01%. How does the minimal SD vary? What are the optimal portfolios?

Efficient Frontier (10 Stocks)
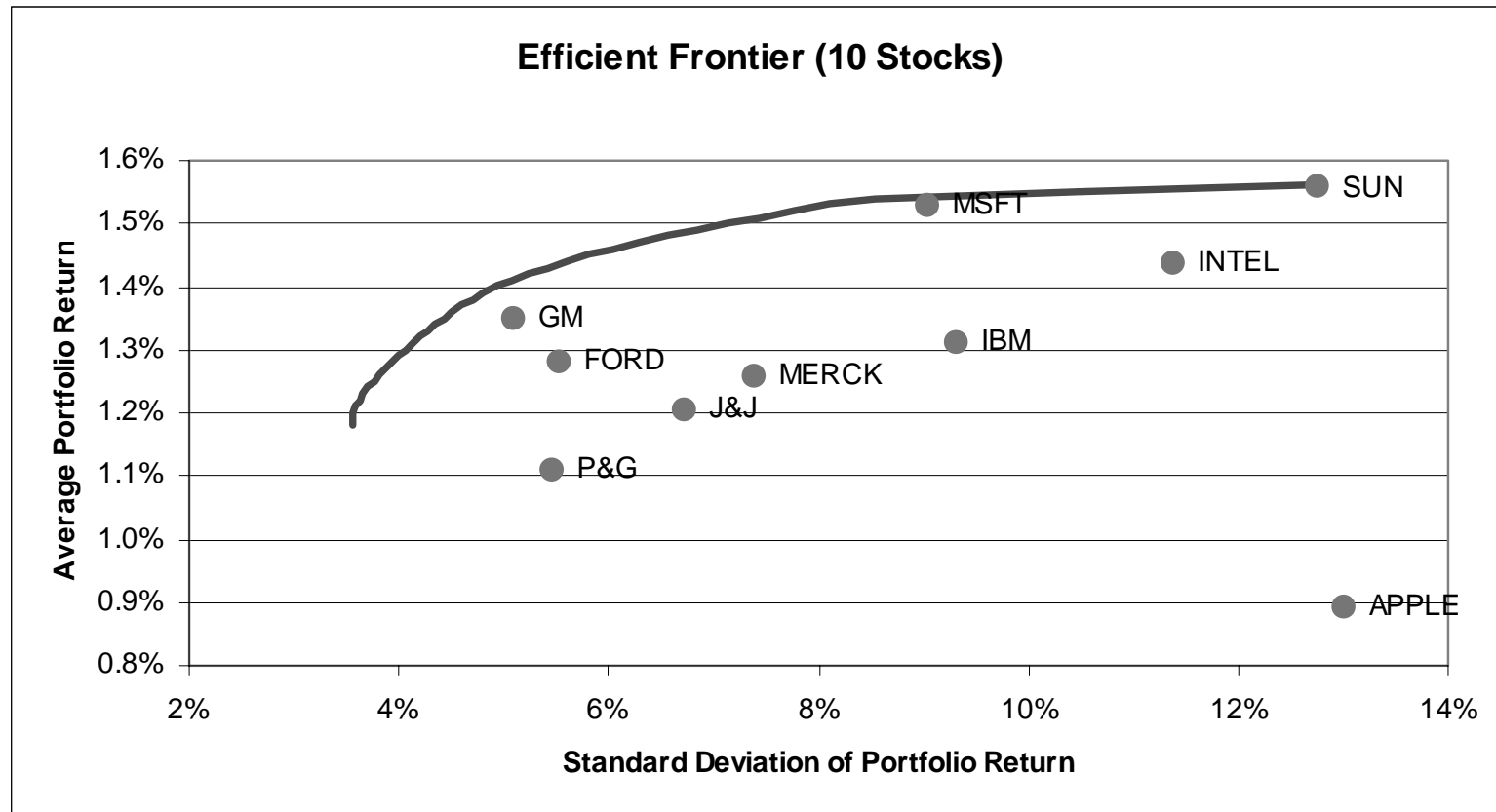
Portfolio Profile as a function of Minimum Mean Return

Portfolio Profile for varying values of the minimum portfolio return
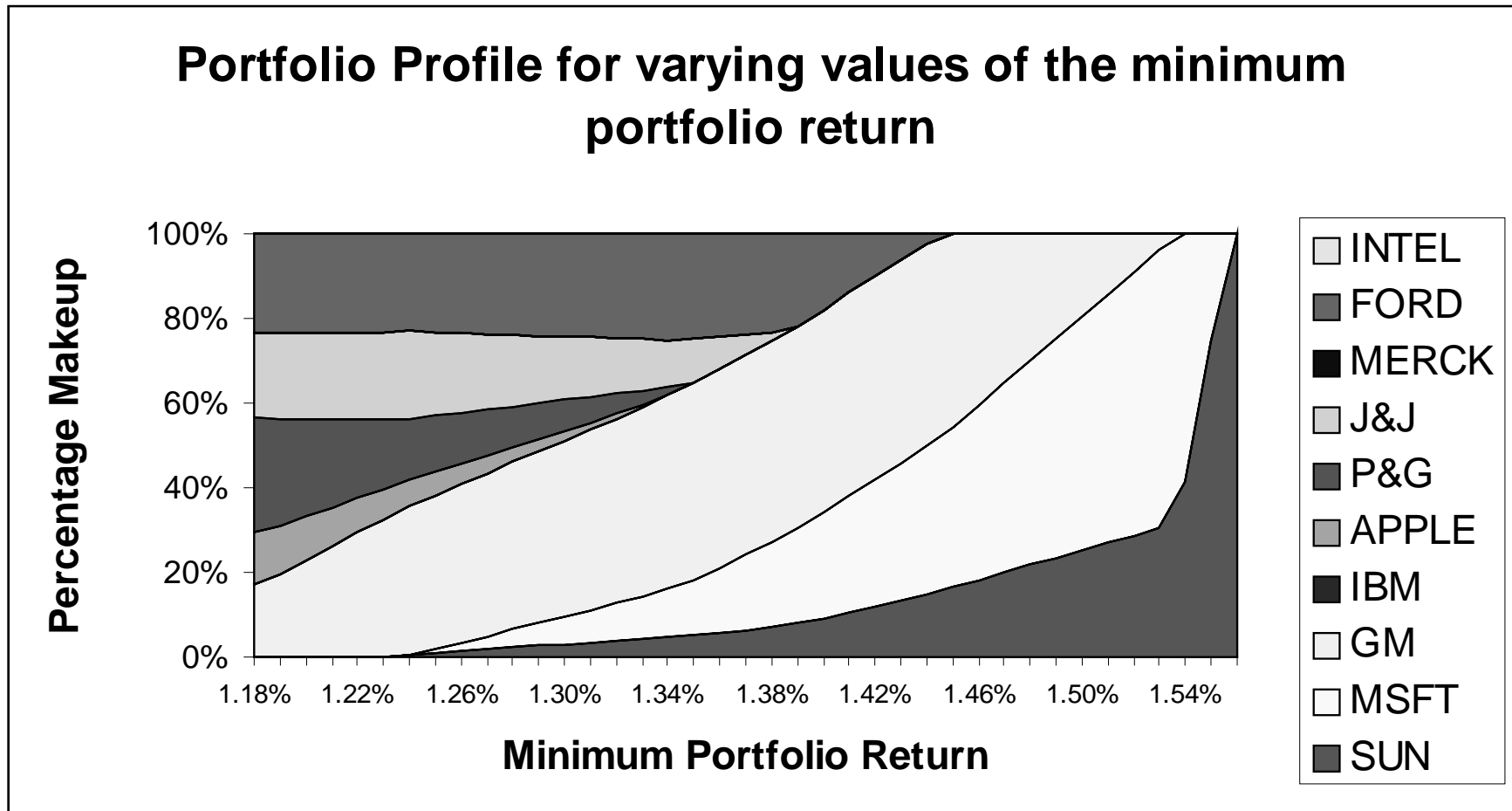
○ This graph demonstrates the makeup of the portfolio as δ (the minimum average-portfolio return) is increased from 1.18% to 1.56%.

Portfolio Optimization (without non-negativity)

○ Consider the same optimization problem, but now without the non-negativity constraints. That is, find the portfolio with the minimum standard deviation of return (SD) that achieves a mean portfolio return of at least 1.35%.

○ Removing the non-negativity constraints allows for *shorting* stocks.

○ What is shorting a stock?

▸ Assume IBM today sells for $160/share and in one month its price is $140/share. During the month, IBM's return was -12.5%.

▸ If you *buy* a share today and sell it one month from now your cash flows are:

Today	A Month From Now
-$160	+$140

▸ If you *short* a share today and "buy" it a month from now, your cash flows are:

| Today	A Month From Now
+$160	-$140

▸ If you short IBM stock during this month, your return is +12.5%.

Optimized Spreadsheet (without non-negativity)

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	TEN-STOCKS.XLS			**Investment Non-Linear Program**											
2										Sum of Portfolio Weights					
3		Avg. Portfolio		Portfolio		Portfolio Weights x(j)				100%	=	100%			
4		Return		Stnd. Dev.		**SUN**	**MSFT**	**GM**	**IBM**	**APPLE**	**P&G**	**J&J**	**MERCK**	**FORD**	**INTEL**
5		1.35%		4.19%		16.9%	28.6%	75.5%	2.3%	6.1%	14.3%	3.7%	-11.6%	-3.7%	-32.2%
6			>=			**SUN**	**MSFT**	**GM**	**IBM**	**APPLE**	**P&G**	**J&J**	**MERCK**	**FORD**	**INTEL**
7	Min Return		1.35%	Mean=		1.6%	1.5%	1.3%	1.3%	0.9%	1.1%	1.2%	1.3%	1.3%	1.4%
8				Stnd. Dev.=		12.8%	9.0%	5.1%	9.3%	13.0%	5.5%	6.7%	7.4%	5.5%	11.4%
9															
10				Adjusted Mean=		1.6%	1.5%	1.3%	1.3%	0.9%	1.1%	1.2%	1.3%	1.3%	1.4%
11															
12		Scen-	Proba-	Portfolio Return		Security Returns by Scenario									
13		ario	bilities	by Scenario		**SUN**	**MSFT**	**GM**	**IBM**	**APPLE**	**P&G**	**J&J**	**MERCK**	**FORD**	**INTEL**
14		1	1/24	1.6%		-0.8%	2.0%	-0.1%	16.1%	-9.7%	-0.6%	11.5%	5.8%	1.1%	-5.7%
15		2	1/24	0.5%		12.5%	3.2%	-2.2%	10.4%	3.2%	-4.2%	-3.4%	-6.6%	4.9%	3.5%
16		3	1/24	2.4%		-18.3%	1.1%	4.3%	-11.9%	-7.0%	1.5%	-2.2%	-7.1%	8.9%	-6.3%
17		4	1/24	2.2%		22.4%	6.4%	2.3%	-5.8%	2.9%	-2.1%	-0.6%	-3.8%	3.3%	16.1%
18		5	1/24	1.9%		13.8%	1.4%	2.0%	-3.5%	10.8%	2.2%	4.4%	5.8%	0.7%	8.4%
19		6	1/24	-4.0%		-7.6%	-2.3%	-4.6%	-9.9%	-16.0%	1.3%	0.8%	-1.0%	-12.4%	-5.8%
20		7	1/24	-7.5%		-8.8%	-5.3%	-6.5%	6.0%	8.4%	-3.3%	-4.4%	-1.6%	-1.1%	-0.7%
21		8	1/24	0.9%		-2.1%	0.5%	2.2%	3.8%	13.9%	-2.2%	2.3%	1.1%	2.4%	3.2%
22		9	1/24	-3.5%		12.6%	4.2%	-2.9%	6.2%	-4.9%	7.9%	3.2%	6.2%	-7.8%	16.6%
23		10	1/24	4.7%		-3.4%	0.6%	12.1%	1.0%	7.3%	-0.3%	-4.7%	3.9%	-1.1%	12.1%
24		11	1/24	5.0%		-6.1%	10.9%	7.8%	20.9%	8.5%	8.0%	7.3%	11.3%	3.7%	12.4%
25		12	1/24	-4.7%		-13.4%	1.9%	-2.9%	-7.6%	-9.8%	-2.8%	-7.4%	-5.1%	-2.6%	0.2%
26		13	1/24	6.4%		22.0%	20.0%	6.2%	0.9%	-16.7%	5.6%	15.2%	12.8%	-1.4%	20.9%
27		14	1/24	0.8%		-4.4%	-7.9%	-1.5%	-11.0%	1.4%	2.1%	-1.3%	0.6%	1.3%	-15.6%
28		15	1/24	-4.5%		-8.1%	-9.4%	-3.9%	-7.1%	16.0%	-6.3%	-8.9%	-9.6%	-5.6%	-5.0%
29		16	1/24	10.2%		-1.8%	29.1%	4.9%	14.3%	-3.2%	7.8%	14.8%	6.2%	9.7%	7.0%
30		17	1/24	3.5%		10.3%	-1.4%	-0.5%	5.2%	1.4%	7.8%	-2.7%	-1.6%	6.9%	-4.1%
31		18	1/24	1.3%		13.8%	-1.5%	-2.4%	1.7%	-10.6%	0.6%	6.5%	12.8%	0.3%	-9.4%
32		19	1/24	8.4%		21.1%	8.5%	11.4%	14.6%	26.5%	5.9%	-4.3%	0.5%	6.5%	26.5%
33		20	1/24	0.5%		4.0%	-10.0%	1.8%	-6.8%	27.9%	-14.3%	-9.6%	-12.6%	4.1%	-2.7%
34		21	1/24	4.0%		-4.6%	-3.3%	7.1%	1.9%	3.4%	2.0%	0.9%	7.8%	3.9%	-2.8%
35		22	1/24	-3.1%		-28.4%	-5.2%	-3.7%	-9.7%	-17.8%	-3.3%	-1.4%	-11.7%	-4.2%	-19.6%
36		23	1/24	1.3%		3.5%	5.4%	-4.7%	8.6%	7.9%	10.2%	8.9%	5.2%	-2.6%	-2.2%
37		24	1/24	4.1%		9.2%	-12.1%	6.2%	-7.1%	-22.4%	3.0%	3.8%	10.8%	11.9%	-12.5%

Solver Options Dialog Box
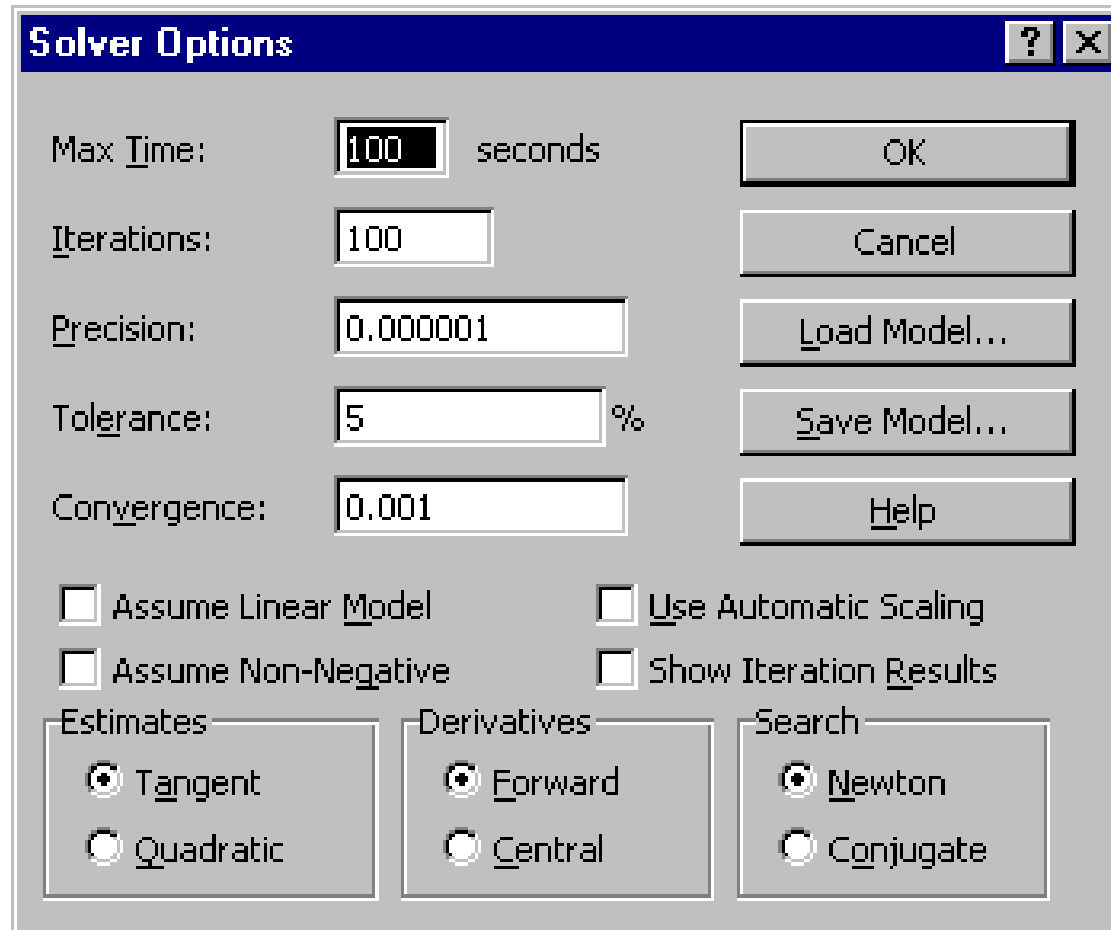
○ Note "Assume Linear Model" is not checked in the Solver *Options* Dialog Box.

Portfolio Optimization (without non-negativity)

❍ The new optimal portfolio has a standard deviation of 4.19%. This is less than the 4.43% we had before (with the non-negativity).

❍ The optimal portfolio has an average portfolio return of 1.35%.

❍ The optimal portfolio is as follows:

▶ **SUN   MSFT   GM   IBM   APPLE**

▶ **16.9%   28.6%   75.5%   2.3%   6.1%**

▶ **P&G   J&J   MERCK   FORD   INTEL**

▶ **14.3%   3.7%   -11.6%   -3.7%   -32.2%**

❍ Comments:

▶ Ford is now shorted, while the previous portfolio was long on Ford.

▶ Intel heavily shorted.

Efficient Frontier (without non-negativity)

❍ Using SolverTable, we can vary the δ (the minimum average return) and trace out an efficient frontier when we allow shorting.

❍ Consider minimizing SD and varying δ from 0% to 3% in increments of 0.1%. How does the efficient frontier with shorting compare to the one without shorting?

Comparison of Efficient Frontiers with and without shorting
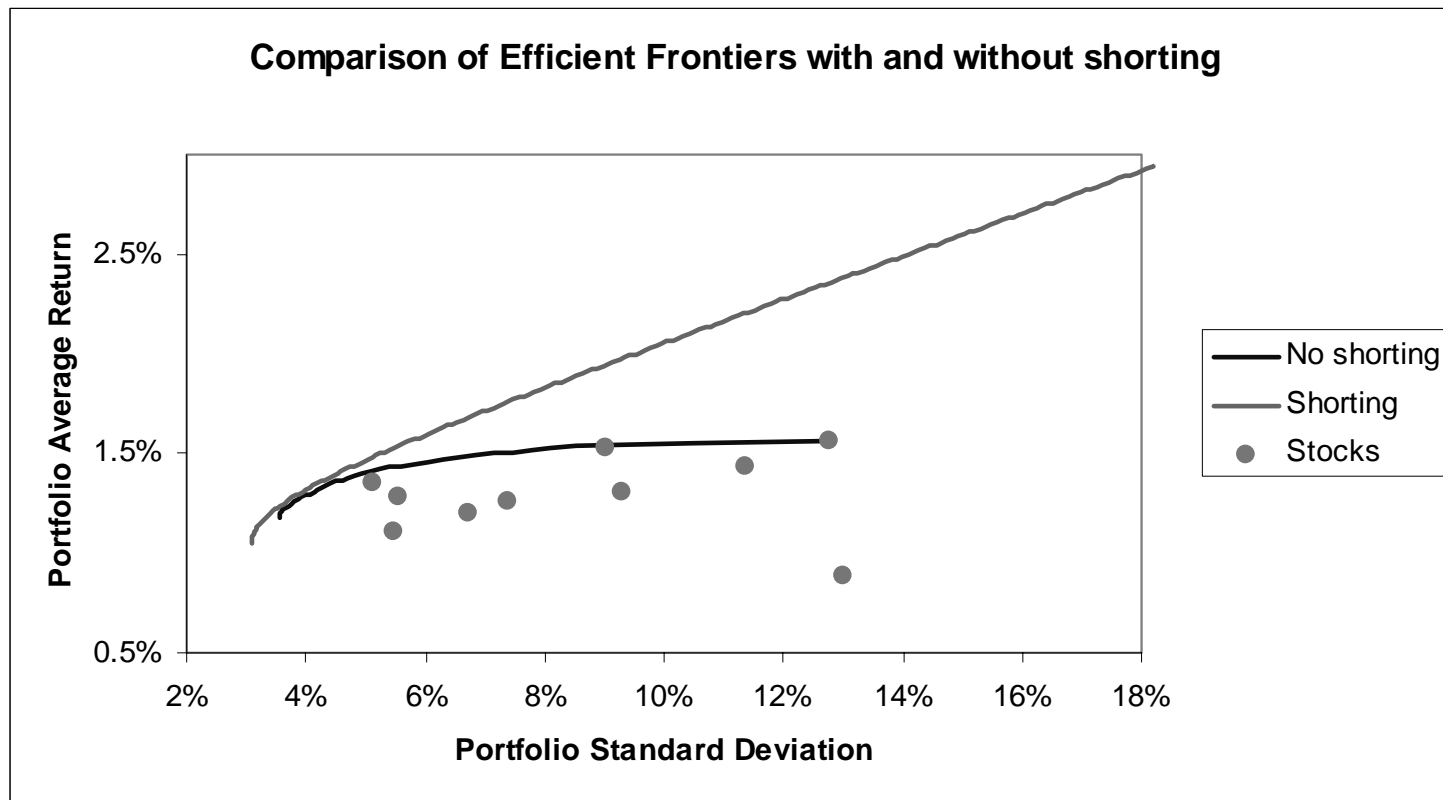
Model Enhancements

❍ The minimum-risk portfolio in our first ten-stock model (w/o short selling) that had an average return of 1.35% was the following:

▸ **SUN   MSFT   GM   J&J   FORD**

5.1%   13.0%   46.5%   10.5%   24.8%

▸ It invests nothing in IBM, APPLE, P&G, Merck or Intel.

❍ The average portfolio return is: 1.35%.

❍ The standard deviation (SD) of the portfolio return is: 4.43%.

❍ The portfolio invested in 5 securities.

❍ What if we wanted to find the minimum-risk portfolio that had at least a 1.35% average return but invested in at most 2 securities.

❍ How could we modify our model to handle that?

Model Enhancements (cont.)

○ Let's go back to our formulation from Lecture 5, which we wrote as follows:

$$\text{min} \quad SD$$

subject to:

$(r_1$ def.$)$	$r_1 = 5.51\ x_1 + 1.95\ x_2 + 2.56\ x_3$
$(r_2$ def.$)$	$r_2 = -1.24\ x_1 + 2.26\ x_2 + 0.16\ x_3$
$(r_3$ def.$)$	$r_3 = 5.46\ x_1 - 4.07\ x_2 - 0.64\ x_3$
$(r_4$ def.$)$	$r_4 = -1.90\ x_1 + 3.59\ x_2 + 0.30\ x_3$
$(r_P$ def.$)$	$r_P = 0.25\ r_1 + 0.25\ r_2 + 0.25\ r_3 + 0.25\ r_4$
(Min. r_P)	$r_P \geq \delta$
(Risk)	$SD = \text{STDEVP}(r_1, r_2, r_3, r_4)$
(Budget)	$x_1 + x_2 + x_3 = 1$
(non-neg.)	$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.$

○ The minimum-risk portfolio that had an average return of at least 1% invested in all 3 securities (x_1=23.2%, x_2=26.4% and x_3=50.4%).

○ Say we want to add a constraint that we can invest in at most 2 securities?

Model Enhancements (cont.)

○ Recall that x_i is the fraction of our fortune that is invested in security *i*.

○ The x_i's satisfy the following constraints:

 ▸ Budget: $x_1 + x_2 + x_3 = 1$

 ▸ Non-neg. $x_1 \geq 0, \ x_2 \geq 0, \ x_3 \geq 0.$

○ We need a way to count the number of securities that a portfolio invests in.

○ To do this we define 3 new integer (*binary*) variables, y_1, y_2 and y_3, one for each security. These variables will be either 0 or 1.

○ Then add the following constraints:

 ▸ $x_1 \leq y_1$

 ▸ $x_2 \leq y_2$

 ▸ $x_3 \leq y_3$

 ▸ $y_1 + y_2 + y_3 \leq 2$

 ▸ $y_1, \ y_2, \ y_3$ binary

○ This will achieve the desired result.

Model Enhancements (cont.)

○ For our ten-stock example, we need to add 10 new binary variables (one per security), a constraint linking x_i and y_i and one constraint limiting the sum of the y-variables.

○ We show the optimized spreadsheet below:

Decision variables (x)

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	TEN-STOCKS.XLS		**Investment Non-Linear Program**												
2									Sum of Portfolio Weights						
3		Avg. Portfolio		Portfolio		Portfolio Weights x(j)				100%	=	100%			
4			Return	Stnd. Dev.		**SUN**	**MSFT**	**GM**	**IBM**	**APPLE**	**P&G**	**J&J**	**MERCK**	**FORD**	**INTEL**
5			1.35%	5.05%	x	0.0%	28.3%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	71.7%	0.0%
6			>=			<=	<=	<=	<=	<=	<=	<=	<=	<=	<=
7	Min Return		1.35%		y	0	1	0	0	0	0	0	0	1	0
8															
9					sum=	2		<=		2 =limit					
10															

=IF(F9<=H9+0.001,"<=", "Not <=")

=SUM(F5:O5)

New Binary Decision Variables (y)
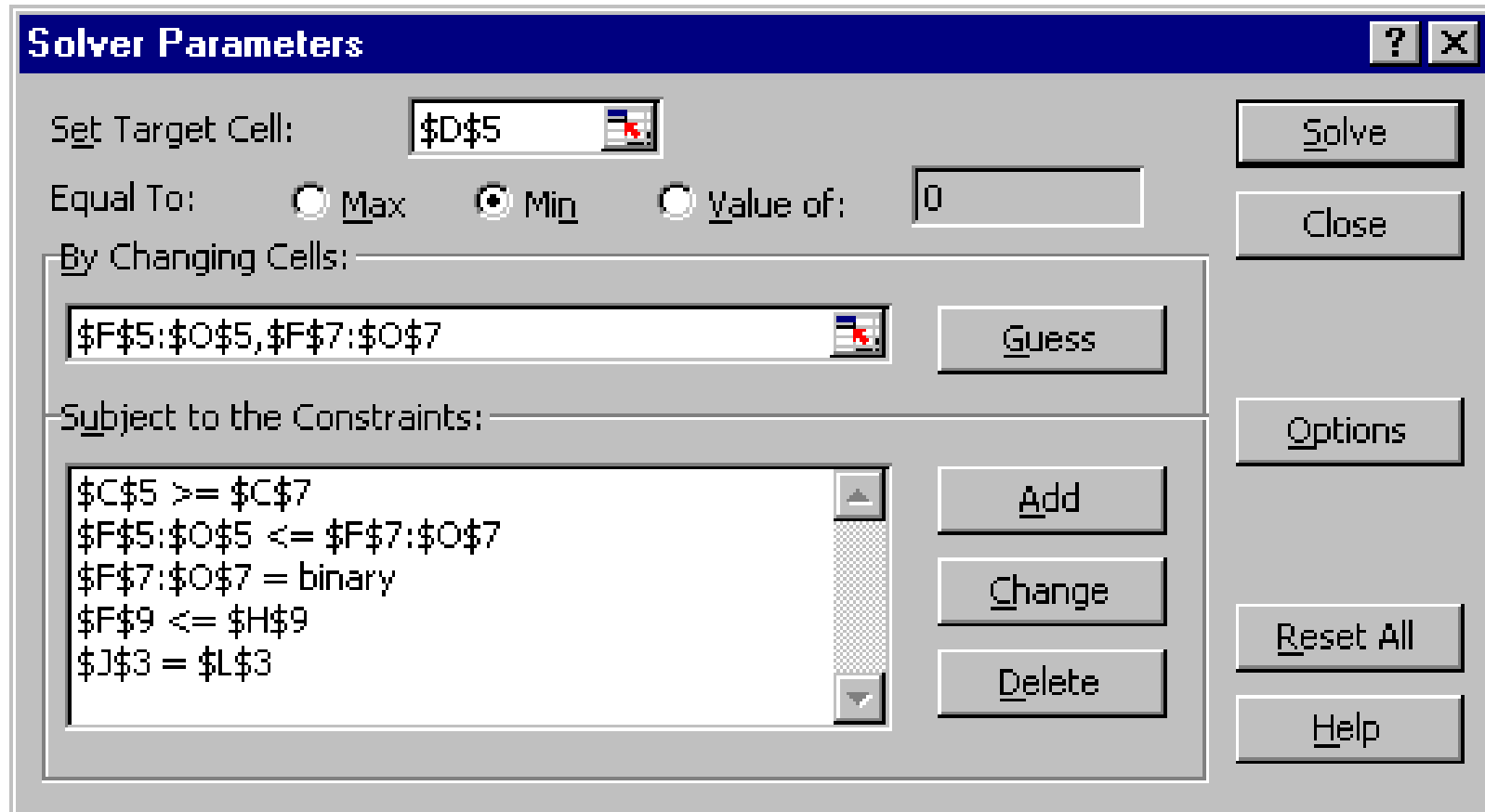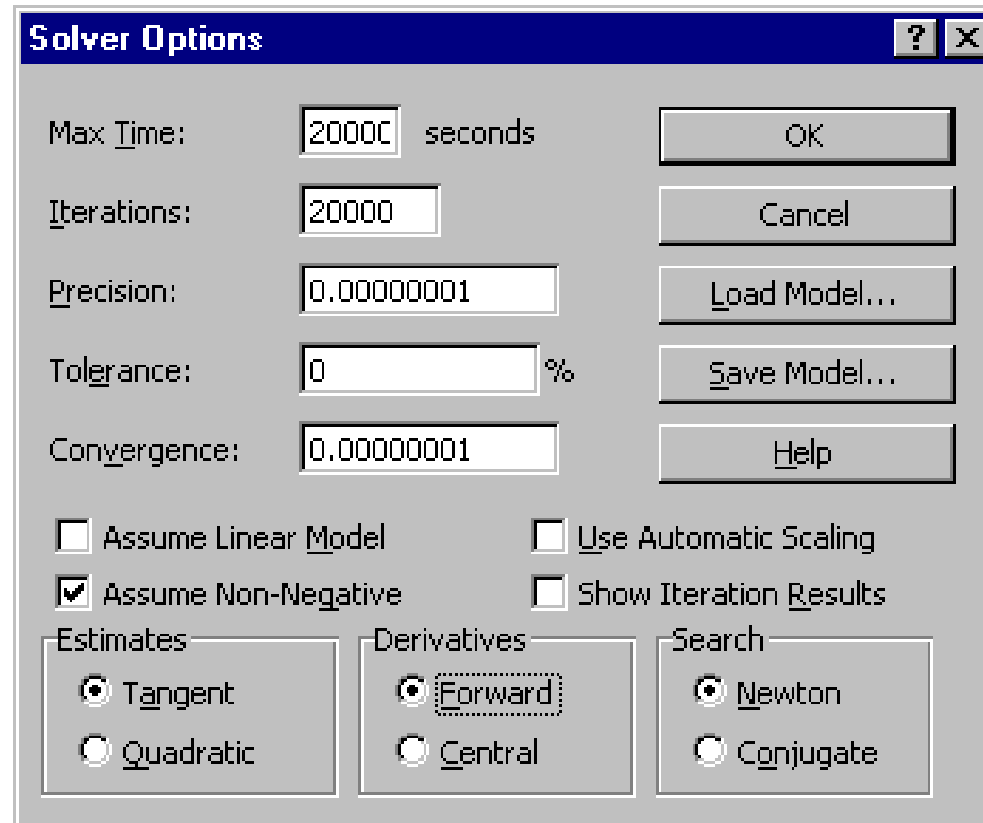
=IF(F4<=F7+0.001,"<=","Not <=")

Model Enhancements (cont.)

The Solver Parameters dialog box.

Model Enhancements (cont.)

The Solver Options dialog box.

❍ Solving problems with binary variables may require giving the computer more time to solve and increasing the precision. Here note that "Tolerance" is set to 0%. It should take more time to solve than a problem without integer variables.

Model Enhancements (cont.)

- ❍ Our new optimal portfolio invests in the following stocks:
  - ▸ **MSFT   FORD**
    - **28.3%   71.7%**
- ❍ The average portfolio return is: 1.35%.
- ❍ The standard deviation (SD) of the portfolio return is: 5.05%.
- ❍ Here is a graph of the efficient frontiers with and without the extra constraint on the number of stocks in the portfolio.
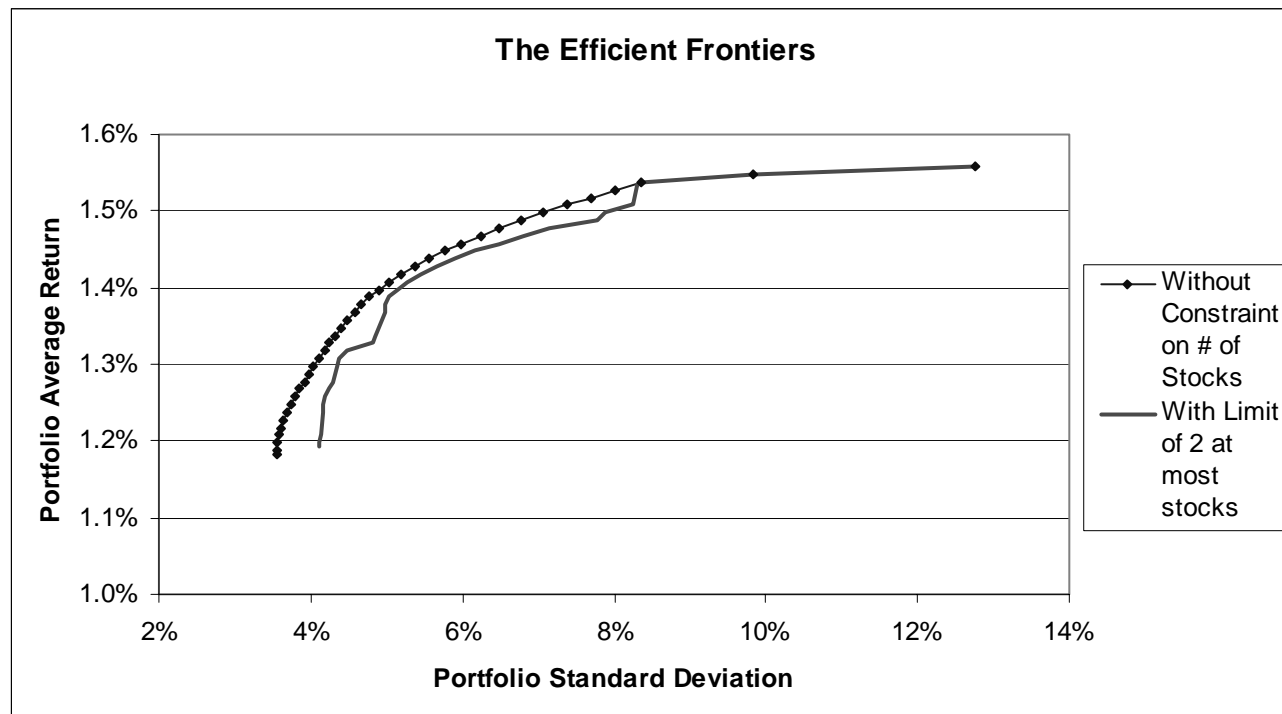
Summary

- ❍ Non-linear programming
- ❍ The effect of correlation on the efficient frontier
- ❍ An example with real data
    - ▸ Adjusting the data to match forecasts
- ❍ Adding a constraint to limit the number of securities in an optimal portfolio

For next class

- ❍ Solve the "GMS Stock Hedging" case, pp.330-331 in the W&A text. (Prepare to discuss the case in class, but do not write up a formal solution.)
- ❍ Read Chapter 7.3 in the W&A text.